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Nambu–Jacobi and generalized Jacobi manifolds

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Abstract. The geometry of Nambu–Jacobi and generalized Jacobi manifolds is studied. A large collection of examples is given. The characteristic distribution generated by the Hamiltonian vector fields on a Nambu–Jacobi manifold is proved to be completely integrable, and the induced geometrical structure of the leaves of the corresponding generalized foliation is ellucidated.

1. Introduction

In [37], Nambu proposed a generalization of Hamiltonian mechanics introducing a n-bracket on $\mathbb{R}^n$ given by

$$\{f_1, \dots, f_n\} = \frac{\partial(f_1 \dots f_n)}{\partial(x^1 \dots x^n)}. \tag{1}$$

This approach was later discussed by Bayen and Flato [8] who investigated the relation of Nambu's mechanics with the Dirac theory of constraints (see also [24, 36, 38]).

In [41], Takhtajan realized that the bracket given by (1) satisfies the so-called fundamental identity

$$\{f_1, \dots, f_{n-1}, \{g_1, \dots, g_n\}\} = \sum_{i=1}^{n} \{g_1, \dots, g_{i-1}, \{f_1, \dots, f_{n-1}, g_i\}, g_{i+1}, \dots, g_n\}. \tag{2}$$

This led him to introduce the notion of Nambu–Poisson manifold as a manifold whose ring of functions is endowed with a n-bracket satisfying (2). However, the fundamental identity was previously considered by Filippov [14] (see also [34, 35]) in a pure algebraic context. Recently, Azcárraga *et al* [2, 3] have considered an alternative identity called the generalized Jacobi identity:

$$\mathrm{Alt}\{f_1, \dots, f_{n-1}, \{g_1, \dots, g_n\}\} = 0 \tag{3}$$

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for $n = 2p$ even. The generalized Jacobi identity led them to introduce the so-called generalized Poisson manifolds. Indeed, the ring of functions of a generalized Poisson manifold carries a n-bracket satisfying (3).

Note that the relation between the ring of functions $C^\infty(M, \mathbb{R})$ and the manifold M is given by

$$\{f_1, \ldots, f_n\} = \Lambda(\mathrm{d}f_1, \ldots, \mathrm{d}f_n)$$

where Λ is a n-vector on M. It should be noted that (2) and (3) are in fact integrability conditions which extend the well known Jacobi identity for Poisson manifolds in the first case, and the vanishing of the Nijenhuis–Schouten bracket of the Poisson tensor with itself in the second case. The relationship between these two kinds of brackets is very simple: any Nambu–Poisson bracket of even order is a generalized Poisson bracket. Furthermore, Azcárraga *et al* have recently proved [5] that if we extend the notion of generalized Poisson manifold by adopting the generalized Jacobi identity (3), where $n = 2p$ is replaced by arbitrary (even or odd) n, then a Nambu–Poisson bracket of arbitrary order is a generalized Poisson bracket. However, it should be noted that in the odd case the generalized Jacobi identity is unrelated to the condition $[\Lambda, \Lambda] = 0$, which is trivially satisfied by any odd-order multivector Λ.

In some sense, Nambu–Poisson manifolds have nicer geometrical properties than generalized Poisson manifolds. In fact, the Hamiltonian vector fields define a generalized foliation such that its leaves inherit a Nambu–Poisson structure. For $n > 3$ the leaves are points or they have dimension n and the induced Nambu–Poisson structure is given by a volume form (i.e. the Nambu–Poisson bracket is locally like (1)). For generalized Poisson manifolds we cannot ensure that the generalized distribution would be involutive in general.

We are assuming that the above brackets are derivations in each argument. However, we can weaken this assumption, and impose that they are only first-order linear n-differential operators. In such a case, if $n = 2$ and we impose the identity (2), or equivalently, the identity (3), we recover the Jacobi manifolds introduced by Lichnerowicz [32] (see also [18, 25]) which, recently, have gained much attention (see [12, 13, 20, 23, 28, 29]). If $n > 2$ and we impose the identity (2) we obtain the notion of Nambu–Jacobi manifolds [34], and if we impose (3) we obtain the notion of generalized Jacobi manifolds [39]. They are the generalizations of Nambu–Poisson and generalized Poisson manifolds, respectively.

The aim of the present paper is to discuss a general geometric framework for Nambu–Jacobi and generalized Jacobi manifolds. To do this, we consider a generalized almost Jacobi bracket of order n on a m-dimensional manifold M which is given by a skew-symmetric first-order linear n-differential operator or equivalently by a pair $(\Lambda, \Box) \in \mathcal{V}^n(M) \oplus \mathcal{V}^{n-1}(M)$, where $\mathcal{V}^r(M)$ is the space of r-vectors on M. The pair $(\Lambda, \Box)$ is called a generalized almost Jacobi structure of order n, and $(M, \Lambda, \Box)$ is called a generalized almost Jacobi manifold. If $\Box = 0$, then we obtain the generalized almost Poisson structures of order n introduced in [19]. By imposing the integrability conditions (2) or (3) we recover the Nambu–Jacobi manifolds discussed in [34] or the generalized Jacobi manifolds studied in [39]. The relation between both is once more simple: any Nambu–Jacobi manifold is generalized Jacobi. We discuss in section 3 a lot of new examples of these kind of geometric structures. Thus, we introduce the notion of compatible Jacobi structures and we prove that if on a manifold M there exist two compatible Jacobi structures then M is a generalized Jacobi manifold of order four. As a consequence, we deduce that the phase space of a time-dependent Hamiltonian system is a generalized Jacobi manifold. Another interesting example of generalized Jacobi manifolds obtained in this section are the 3-Sasakian manifolds. These manifolds appear in many places in geometry and mathematical physics. In fact, in [15], the authors showed that a 3-Sasakian structure on a manifold of dimension seven is characterized by the existence of at least three independent Killing spinors (see also [7]). Also, in [6], the relation between the 3-Sasakian structures on the unit sphere S^7 and the instantons is discussed. More examples of generalized Jacobi manifolds are the unit spheres of certain types of Euclidean vector spaces. More precisely, we prove that if $\mathfrak{g}$ is a generalized Lie algebra of order $2n$ endowed with an inner product then the unit sphere of $\mathfrak{g}$ is a generalized Jacobi manifold of order $2n$. In particular, for each primitive invariant symmetric polynomial K on a simple compact Lie algebra $\mathfrak{g}$, we obtain a generalized Jacobi structure on the unit sphere of $\mathfrak{g}$. To do this, we use the structure of generalized Lie algebra on $\mathfrak{g}$ defined by K (see [3]). Examples of Nambu–Jacobi manifolds are also given in this section. These examples are fundamental in the sense that any Nambu–Jacobi manifold is constructed by gluing pieces like them. Indeed, in section 4, after introducing the notion of a Hamiltonian vector field on a generalized almost Jacobi manifold, we prove that the characteristic distribution generated by the Hamiltonian vector fields on a Nambu–Jacobi manifold is involutive (theorem 4.3), and we obtain that each leaf of this generalized foliation inherits a Nambu–Jacobi structure. These results globalize the local ones given by Marmo *et al* in [34].

2. Nambu–Jacobi and generalized Jacobi manifolds

Let M be a differentiable manifold of dimension m. We will use the following notation:

- $C^\infty(M,\mathbb{R})$ is the algebra of C^∞ real-valued functions on M;
- $\mathfrak{X}(M)$ is the $C^\infty(M,\mathbb{R})$-module of vector fields on M;
- $\mathcal{V}^k(M)$ is the space of k-vectors on M;
- $\Omega^k(M)$ is the space of k-forms on M.

Moreover, our conventions for the exterior calculus are those of [43].

A generalized almost Jacobi bracket of order n on M is an n-linear mapping

$$\{,\ldots,\}: C^\infty(M,\mathbb{R})\times\ldots^{(n}\ldots\times C^\infty(M,\mathbb{R})\longrightarrow C^\infty(M,\mathbb{R})$$

satisfying the following properties.

(i) *Skew-symmetry:*

$$\{f_1,\ldots,f_n\}=\varepsilon_\sigma\{f_{\sigma(1)},\ldots,f_{\sigma(n)}\}$$

for all $f_1,\ldots,f_n\in C^\infty(M,\mathbb{R})$ and $\sigma\in\mathfrak{S}_n$, where $\mathfrak{S}_n$ is the group of the permutations of order n and ε_σ is the signature of the permutation σ.

(ii) $\{,\ldots,\}$ is a first-order linear differential operator on M with respect to each argument, i.e.

$$\{f_1g_1,f_2,\ldots,f_n\}=f_1\{g_1,f_2,\ldots,f_n\}+g_1\{f_1,\ldots,f_n\}-f_1g_1\{1,f_2,\ldots,f_n\}$$

for all $f_1,g_1,f_2,\ldots,f_n\in C^\infty(M,\mathbb{R})$.

If $\{,\ldots,\}$ is a generalized almost Jacobi bracket of order n then we can define an n-vector Λ and an $(n-1)$-vector $\Box$ as follows

$$\Box(\mathrm{d}f_1,\ldots,\mathrm{d}f_{n-1})=\{1,f_1,\ldots,f_{n-1}\}$$

$$\Lambda(\mathrm{d}f_1,\ldots,\mathrm{d}f_n)=\{f_1,\ldots,f_n\}+\sum_{i=1}^{n}(-1)^i f_i\Box(\mathrm{d}f_1,\ldots,\widehat{\mathrm{d}f_i},\ldots,\mathrm{d}f_n)$$

for all $f_1, \ldots, f_n \in C^\infty(M, \mathbb{R})$. Conversely, any pair $(\Lambda, \Box) \in \mathcal{V}^n(M) \oplus \mathcal{V}^{n-1}(M)$ defines a *generalized almost Jacobi bracket of order* n given by

$$\{f_1, \ldots, f_n\} = \Lambda(\mathrm{d}f_1, \ldots, \mathrm{d}f_n) + \sum_{i=1}^{n} (-1)^{i-1} f_i \Box(\mathrm{d}f_1, \ldots, \widehat{\mathrm{d}f_i}, \ldots, \mathrm{d}f_n).$$

Thus, the pair $(\Lambda, \Box) \in \mathcal{V}^n(M) \oplus \mathcal{V}^{n-1}(M)$ is called a *generalized almost Jacobi structure of order* n and $(M, \Lambda, \Box)$ is a *generalized almost Jacobi manifold*. If $\Box = 0$, then we obtain a *generalized almost Poisson structure of order* n (see [19]).

A more rich structure can be obtained by adding integrability conditions to the associated generalized almost Jacobi bracket $\{, \ldots, \}$. In fact, two different integrability conditions may be assumed.

(iii$_1$) (*Fundamental identity*)

$$\{f_1, \ldots, f_{n-1}, \{g_1, \ldots, g_n\}\} = \sum_{i=1}^{n} \{g_1, \ldots, g_{i-1}, \{f_1, \ldots, f_{n-1}, g_i\}, g_{i+1}, \ldots, g_n\} \tag{4}$$

for all $f_1, \ldots, f_{n-1}, g_1, \ldots, g_n \in C^\infty(M, \mathbb{R})$. In this case, $\{, \ldots, \}$ is called a *Nambu–Jacobi* or *n-Jacobi bracket*, and $(M, \Lambda, \Box)$ is a *Nambu–Jacobi manifold of order* n (see [34]). Note that if $\Box = 0$, $\{, \ldots, \}$ is a derivation in each argument and, therefore, it defines a *Nambu–Poisson bracket*, and (M, Λ) is a *Nambu–Poisson manifold of order* n (see [41]).

In [34], the authors have proved the following results.

Proposition 2.1. Let $(M, \Lambda, \Box)$ be a Nambu–Jacobi manifold of order n, $n > 2$. Then

(i) Λ is a Nambu–Poisson structure on M of order n.

(ii) $\Box$ is a Nambu–Poisson structure on M of order $n - 1$.

(iii) For every $x \in M$ such that $\Lambda(x) \neq 0$, there exists $\theta_x \in T_x^*M$ such that $i_{\theta_x}\Lambda(x) = \Box(x)$.

(iv) If $f_1, \ldots, f_{n-2} \in C^\infty(M, \mathbb{R})$ we have

$$\mathcal{L}_{X^{\Box}_{f_1 \cdots f_{n-2}}} \Lambda = 0$$

where $\mathcal{L}$ is the Lie derivative operator on M and $X^{\Box}_{f_1 \cdots f_{n-2}}$ is the vector field defined by

$$X^{\Box}_{f_1 \cdots f_{n-2}}(h) = \Box(\mathrm{d}f_1, \ldots, \mathrm{d}f_{n-2}, \mathrm{d}h) \qquad \text{for all } h \in C^\infty(M, \mathbb{R}).$$

Remark 2.2. Let Λ be a generalized almost Poisson structure of order n on a manifold M and $f_1, \ldots, f_{n-1} \in C^\infty(M, \mathbb{R})$. Then, the vector field $X^{\Lambda}_{f_1 \cdots f_{n-1}}$ given by

$$X^{\Lambda}_{f_1 \cdots f_{n-1}}(h) = \Lambda(\mathrm{d}f_1, \ldots, \mathrm{d}f_{n-1}, \mathrm{d}h)$$

for all $h \in C^\infty(M, \mathbb{R})$ is called the *Hamiltonian vector field* associated with the functions $f_1, \ldots, f_{n-1}$ (see [19]).

(iii$_2$) (*Generalized Jacobi identity*)

$$\mathrm{Alt}\{f_1, \ldots, f_{n-1}, \{g_1, \ldots, g_n\}\} = 0 \tag{5}$$

for all $f_1, \ldots, f_{n-1}, g_1, \ldots, g_n \in C^\infty(M, \mathbb{R})$. In this case, $\{, \ldots, \}$ is called a *generalized Jacobi bracket*, and $(M, \Lambda, \Box)$ is a *generalized Jacobi manifold of order* n (see [39]). If $\Box = 0$ then $\{, \ldots, \}$ is a derivation in each argument and (M, Λ) is a generalized Poisson manifold (see [2, 3, 5, 19]).

For n even, the integrability conditions of the generalized Jacobi structure are given by the following result (see [39]).

Proposition 2.3. Let $(M, \Lambda, \Box)$ be a generalized almost Jacobi manifold of even order n, $n = 2s$. Then $(M, \Lambda, \Box)$ is a generalized Jacobi manifold if and only if

$$[\Lambda, \Lambda] = 2(2s - 1)\Lambda \wedge \Box \qquad \text{and} \qquad [\Lambda, \Box] = 0$$

where [,] denotes the Schouten–Nijenhuis bracket.

If $n = 2$ then the fundamental identity and the generalized Jacobi identity coincide and, in such a case, $(M, \Lambda, \Box)$ is a Jacobi manifold [32]. Interesting examples of Jacobi manifolds are contact manifolds and locally conformal symplectic manifolds which we will describe below.

Let M be a $(2m + 1)$-dimensional manifold and η a 1-form on M. We say that η is a contact 1-form if $\eta \wedge (\mathrm{d}\eta)^m \neq 0$ at every point. In such a case (M, η) is called a *contact manifold* (see, for example, [9]). The Darboux theorem ([9]) states that around every point of M there exist canonical coordinates $(t, q^1, \ldots, q^m, p_1, \ldots, p_m)$ such that

$$\eta = \mathrm{d}t - \sum_{i=1}^{m} p_i \, \mathrm{d}q^i .$$

A contact manifold (M, η) is a Jacobi manifold. In fact, we define the 2-vector Λ on M by

$$\Lambda(\alpha, \beta) = \mathrm{d}\eta(\flat^{-1}(\alpha), \flat^{-1}(\beta)) \tag{6}$$

for all $\alpha, \beta \in \Omega^1(M)$, where $\flat : \mathfrak{X}(M) \longrightarrow \Omega^1(M)$ is the isomorphism of $C^\infty(M, \mathbb{R})$-modules given by $\flat(X) = i_X \, \mathrm{d}\eta + \eta(X)\eta$. The vector field $\Box$ is just the Reeb vector field $\Box = \flat^{-1}(\eta)$ of (M, η). We remark that $i_\Box \eta = 1$ and $i_\Box \, \mathrm{d}\eta = 0$. Using canonical coordinates we get

$$\Lambda = \sum_{i=1}^{m} \left(\frac{\partial}{\partial q^i} + p_i \frac{\partial}{\partial t} \right) \wedge \frac{\partial}{\partial p_i} \qquad \Box = \frac{\partial}{\partial t}.$$

An interesting class of contact manifolds are the Sasakian manifolds which we will describe next.

Let $(M, \varphi, \Box, \eta, g)$ be a $(2m + 1)$-dimensional *almost contact metric manifold*, that is (see [9]), φ is a (1,1) tensor field, η is a 1-form, $\Box$ is a vector field and g is a Riemannian metric on M such that

$$\varphi^2 = -\mathrm{Id} + \eta \otimes \Box \qquad \eta(\Box) = 1 \qquad g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y)$$

for all $X, Y \in \mathfrak{X}(M)$, Id being the identity transformation. Then we have $\varphi(\Box) = 0$, $\eta \circ \varphi = 0$ and $\eta(X) = g(X, \Box)$, for all $X \in \mathfrak{X}(M)$. The *fundamental* 2*-form* Φ of M is defined by $\Phi(X, Y) = g(X, \varphi Y)$, and the $(2m + 1)$-form $\eta \wedge \Phi^m$ is a volume form on M. The almost contact metric is said to be ([10]): *normal* if $[\varphi, \varphi] + \mathrm{d}\eta \otimes \Box = 0$ and *Sasakian* if it is normal and $\mathrm{d}\eta = 2\Phi$. If $(M, \varphi, \Box, \eta, g)$ is a Sasakian manifold then (M, η) is a contact manifold.

Other examples of Jacobi manifolds are the locally conformal symplectic manifolds.

An *almost symplectic manifold* is a pair (M, Ω), where M is an even-dimensional manifold and Ω is a non-degenerate 2-form on M. An almost symplectic manifold is said to be *locally conformal symplectic (l.c.s.)* if for each point $x \in M$ there is an open neighbourhood U such that $\mathrm{d}(\mathrm{e}^{-\sigma}\Omega) = 0$, for some function $\sigma : U \longrightarrow \mathbb{R}$. If $U = M$ then M is said to be a *globally conformal symplectic (g.c.s.)* manifold (see, for example, [42]). An almost symplectic manifold (M, Ω) is l.(g.)c.s. if and only if there exists a closed (exact) 1-form ω such that

$$\mathrm{d}\Omega = \omega \wedge \Omega. \tag{7}$$

The 1-form ω is called the *Lee 1-form* of M. It is obvious that the l.c.s. manifolds with Lee 1-form identically zero are just the symplectic manifolds. We define a 2-vector Λ and a vector field $\Box$ by

$$\Lambda(\alpha,\beta)=\Omega(\flat^{-1}(\alpha),\flat^{-1}(\beta)) \qquad \Box=\flat^{-1}(\omega) \tag{8}$$

for all $\alpha,\beta\in\Omega^1(M)$, where $\flat:\mathfrak{X}(M)\longrightarrow\Omega^1(M)$ is the isomorphism of $C^\infty(M,\mathbb{R})$-modules given by $\flat(X)=i_X\Omega$. Then $(M,\Lambda,\Box)$ is a Jacobi manifold.

Moreover, around every point of M there exist canonical coordinates $(q^1,\ldots,q^m,p_1,\ldots,p_m)$ and a local differentiable function σ such that

$$\Omega=\mathrm{e}^{\sigma}\sum_i \mathrm{d}q^i\wedge\mathrm{d}p_i \qquad \omega=\mathrm{d}\sigma=\sum_i\left(\frac{\partial\sigma}{\partial q^i}dq^i+\frac{\partial\sigma}{\partial p_i}dp_i\right)$$

$$\Lambda=\mathrm{e}^{-\sigma}\sum_i\left(\frac{\partial}{\partial q^i}\wedge\frac{\partial}{\partial p_i}\right) \qquad \Box=\mathrm{e}^{-\sigma}\sum_i\left(\frac{\partial\sigma}{\partial p_i}\frac{\partial}{\partial q^i}-\frac{\partial\sigma}{\partial q^i}\frac{\partial}{\partial p_i}\right).$$

Finally, a very simple but interesting Jacobi structure is that provided by a vector field $\Box$ on a manifold M, i.e. the Jacobi structure is given by $(\Lambda=0,\Box)$. This structure is closely related with Virasoro algebras (see [17]).

Now, let $(M,\Lambda,\Box)$ be a generalized almost Jacobi manifold of even order n, $n=2s$. For each $p\in\mathbb{R}-\{0\}$, we define on $M\times\mathbb{R}$ the $2s$-vector

$$(\widetilde{\Lambda,\Box})_p=\mathrm{e}^{-pt}\Lambda+\mathrm{e}^{-pt}\frac{\partial}{\partial t}\wedge\Box \tag{9}$$

where t is the usual coordinate on $\mathbb{R}$. Then, from proposition 2.3 and the properties of the Schouten–Nijenhuis bracket, we conclude the following.

Proposition 2.4. $(\Lambda,(p/(2s-1))\Box)$ is a generalized Jacobi structure on M if and only if $(\widetilde{\Lambda,\Box})_p$ is a generalized Poisson structure on $M\times\mathbb{R}$.

Remark 2.5. If $(M,\Lambda,\Box)$ is a Jacobi manifold then the Poisson manifold $(M\times\mathbb{R},(\widetilde{\Lambda,\Box})_1)$ is called the Poissonization of the Jacobi manifold M (see [20]) or the tangentially exact Poisson manifold associated with M (see [32]). For a contact manifold (M,η), $(\widetilde{\Lambda,\Box})_1$ is just its symplectification (see [31]), i.e. the Poisson structure defined by the symplectic form

$$\Omega=\mathrm{e}^t\,\mathrm{d}\eta+\mathrm{e}^t\,\mathrm{d}t\wedge\eta. \tag{10}$$

In particular, when $(M,\varphi,\Box,\eta,g)$ is a Sasakian manifold we can define on the product manifold $M\times\mathbb{R}$ a complex structure J as follows (see [9]):

$$J=\varphi\circ(\mathrm{pr}_1)_*-\frac{1}{2}(\mathrm{pr}_2)^*(\mathrm{d}t)\otimes\Box+2(\mathrm{pr}_1)^*\eta\otimes\frac{\partial}{\partial t}$$

where $\mathrm{pr}_1:M\times\mathbb{R}\longrightarrow M$ and $\mathrm{pr}_2:M\times\mathbb{R}\longrightarrow\mathbb{R}$ are the canonical projections onto the first and second factor, respectively. Moreover, the Riemannian metric on $M\times\mathbb{R}$ defined by

$$h=\mathrm{e}^t(2\mathrm{pr}_1^*(g)+\tfrac{1}{2}\mathrm{pr}_2^*(\mathrm{d}t)\otimes\mathrm{pr}_2^*(\mathrm{d}t))$$

is compatible with J and a direct computation proves that the Kähler 2-form of the Hermitian structure (J,h) is just the symplectic 2-form Ω given by (10). Therefore, we conclude that the symplectification of a Sasakian manifold is a Kähler manifold. Remember that the Kähler 2-form of a Hermitian structure (J,h) on a manifold N is the 2-form Ω defined by $\Omega(X,Y)=h(X,JY)$, and that N is said to be Kähler if Ω is closed.

The relationship between Nambu–Jacobi and generalized Jacobi manifolds is given in the following result

Proposition 2.6. Every Nambu–Jacobi manifold is a generalized Jacobi manifold.

Proof. In fact, let M be a Nambu–Jacobi manifold with Nambu–Jacobi bracket $\{,\ldots,\}$. Using (4) and the antisymmetry of the bracket $\{,\ldots,\}$, we obtain

$$\mathrm{Alt}\{f_1,\ldots,f_{n-1},\{g_1,\ldots,g_n\}\} = n(-1)^{n-1}\mathrm{Alt}\{f_1,\ldots,f_{n-1},\{g_1,\ldots,g_n\}\}$$

which implies that $\mathrm{Alt}\{f_1,\ldots,f_{n-1},\{g_1,\ldots,g_n\}\} = 0$. □

3. Examples of Nambu–Jacobi and generalized Jacobi manifolds

In this section, we will describe some examples of Nambu–Jacobi and generalized Jacobi manifolds.

First, we will give some examples of Nambu–Jacobi manifolds. Some of these examples have been obtained in [34].

Example 1. Let (M,Λ) be a Nambu–Poisson manifold of order n, $n > 2$. If $f \in C^\infty(M,\mathbb{R})$ then $i_{\mathrm{d}f}\Lambda$ is also a Nambu–Poisson structure on M (see [19]).

On the other hand, since Λ is locally decomposable (see [16]) then we have

$$X^{\Lambda}_{f_1\ldots f_{n-1}} \wedge \Lambda = 0 \tag{11}$$

for all $f_1,\ldots,f_{n-1}$, where $X^{\Lambda}_{f_1\ldots f_{n-1}}$ is the Hamiltonian vector field on (M,Λ) associated with the functions $f_1,\ldots,f_{n-1}$. Thus, from (4) and (11), we obtain that $(\Lambda, i_{\mathrm{d}f}\Lambda)$ is a Nambu–Jacobi structure of order n.

Since a closed 1-form is locally exact, the above result can be generalized as follows.

Proposition 3.1. If (M,Λ) is a Nambu–Poisson manifold of order n, $n > 2$, and θ is a closed 1-form on M then $(\Lambda, i_\theta\Lambda)$ is a Nambu–Jacobi structure of order n on M.

Next, let M be an oriented manifold of dimension n, and choose a volume form ν on M. Denote by Λ_ν the n-vector on M given by

$$\Lambda_\nu(\mathrm{d}f_1,\ldots,\mathrm{d}f_n)\nu = \mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_n \tag{12}$$

for all $f_1,\ldots,f_n \in C^\infty(M,\mathbb{R})$. Then (M,Λ_ν) is a Nambu–Poisson manifold of order n (see [16] and [19]). Moreover, we have the following.

Proposition 3.2. Let Λ_ν be the Nambu–Poisson tensor of order n associated with a volume form ν on an oriented manifold M of dimension n. Suppose that θ is a 1-form on M. Then, the generalized almost Jacobi manifold $(M,\Lambda_\nu, i_\theta\Lambda_\nu)$ is a Nambu–Jacobi manifold if and only if θ is a closed 1-form.

Proof. If θ is a closed 1-form on M, it directly follows from proposition 3.1 that $(\Lambda_\nu, i_\theta\Lambda_\nu)$ is a Nambu–Jacobi structure on M.

Conversely, suppose that θ is a 1-form on M such that $(M,\Lambda_\nu, i_\theta\Lambda_\nu)$ is a Nambu–Jacobi manifold of order n. From (12), we deduce that there exist local coordinates $(x_1,\ldots,x_n)$ such that

$$\Lambda_\nu = \frac{\partial}{\partial x_1} \wedge \cdots \wedge \frac{\partial}{\partial x_n}.$$

Moreover, if $\theta = \theta^k \,\mathrm{d}x_k$ then, from proposition 2.1 (iv) and using the Hamiltonian vector field

$$X^{i_\theta \Lambda_\nu}_{x_1 \ldots \widehat{x_i} \ldots \widehat{x_j} \ldots x_n} = (-1)^{n+i+j} \left(\theta^j \frac{\partial}{\partial x_i} - \theta^i \frac{\partial}{\partial x_j} \right)$$

we obtain that

$$\left(\frac{\partial \theta^j}{\partial x_i} - \frac{\partial \theta^i}{\partial x_j} \right) \Lambda_\nu = 0$$

for all $i, j = 1, \ldots, n$. Hence, $\mathrm{d}\theta = 0$. □

Example 2. Let (M, Λ) be a Nambu–Poisson manifold of order n, $n > 2$ (respectively, a symplectic manifold of dimension two). Proceeding as in example 1, and using (4) and the fact that Λ is locally decomposable, we deduce that the pair $(0, \Lambda)$ is a Nambu–Jacobi structure of order $n + 1$ (respectively, of order three) on M.

Next, we will give some examples of generalized Jacobi manifolds.

Example 3. Let $\Box$ be a $(2n - 1)$-vector on a manifold M. Then, from proposition 2.3 we deduce that $(0, \Box)$ is a generalized Jacobi structure of order $2n$ on M.

Example 4. Let Λ be a generalized Poisson structure on M of order $2n$. If $f \in C^\infty(M, \mathbb{R})$ then

$$[f\Lambda, f\Lambda] = 2i_{\mathrm{d}f}\Lambda \wedge (f\Lambda). \tag{13}$$

On the other hand, we can define a homomorphism of $C^\infty(M, \mathbb{R})$-modules $\# : \Omega^{2n-1}(M) \longrightarrow \mathfrak{X}(M)$ as follows:

$$\#(\alpha_1 \wedge \cdots \wedge \alpha_{2n-1})(\beta) = \Lambda(\alpha_1, \ldots, \alpha_{2n-1}, \beta) \text{ for } \alpha_1, \ldots, \alpha_{2n-1}, \beta \in \Omega^1(M).$$

Now, consider the homomorphism of $C^\infty(M, \mathbb{R})$-modules $\tilde{\#} : \Omega^k(M) \longrightarrow \mathcal{V}^{k(2n-1)}(M)$ given by

$$\tilde{\#}(\alpha)(\alpha_1, \ldots, \alpha_{k(2n-1)}) = \frac{(-1)^k}{k!((2n-1)!)^k} \sum_{\sigma \in \mathfrak{S}_{k(2n-1)}} \varepsilon_\sigma \alpha(\#(\alpha_{\sigma(1)} \wedge \cdots \wedge \alpha_{\sigma(2n-1)}),$$
$$\times \ldots, \#(\alpha_{\sigma((k-1)(2n-1)+1)} \wedge \cdots \wedge \alpha_{\sigma(k(2n-1))}))$$

for all $\alpha \in \Omega^k(M)$ and $\alpha_1, \ldots, \alpha_{k(2n-1)} \in \Omega^1(M)$, where $\mathfrak{S}_{k(2n-1)}$ is the group of the permutations of order $k(2n - 1)$ and ε_σ is the signature of the permutation σ (see [22]).

Then we have $\tilde{\#}(\mathrm{d}\alpha) = [\Lambda, \tilde{\#}\alpha]$, for all $\alpha \in \Omega^k(M)$ (see [22] for a proof). Taking $\alpha = \mathrm{d}f$ we conclude that

$$[\Lambda, i_{\mathrm{d}f}\Lambda] = 0. \tag{14}$$

Thus, using (13), (14) and proposition 2.3, we deduce that $(f\Lambda, (1/(2n - 1))i_{\mathrm{d}f}\Lambda)$ is a generalized Jacobi structure of order $2n$ on M.

Remark 3.3. If Λ is a symplectic structure and $f = \mathrm{e}^{-\sigma}$ is a positive function then the corresponding Jacobi structure $(f\Lambda, i_{\mathrm{d}f}\Lambda)$ is g.c.s. (see section 2).

Example 5. (*Bihamiltonian manifolds*) First, we will introduce the notion of compatible Jacobi structures. This definition extends the definition of compatible Poisson structures.

Two Jacobi structures $(\Lambda_1, \Box_1)$ and $(\Lambda_2, \Box_2)$ on a manifold M are said to be *compatible* if $(\Lambda_1 + \Lambda_2, \Box_1 + \Box_2)$ is a Jacobi structure on M.

A direct computation, using (9), proposition 2.3 and remark 2.5, shows the following result.

Proposition 3.4. Let $(\Lambda_1, \Box_1)$ and $(\Lambda_2, \Box_2)$ be two Jacobi structures on a manifold M. Then the following statements are equivalent:

(i) the structures $(\Lambda_1, \Box_1)$ and $(\Lambda_2, \Box_2)$ are compatible;

(ii) $[\Lambda_1, \Lambda_2] = \Box_1 \wedge \Lambda_2 + \Box_2 \wedge \Lambda_1$ and $\mathcal{L}_{\Box_1}\Lambda_2 + \mathcal{L}_{\Box_2}\Lambda_1 = 0$;

(iii) the Poissonizations

$$(\widetilde{\Lambda_1, \Box_1})_1 = \mathrm{e}^{-t}\Lambda_1 + \mathrm{e}^{-t}\frac{\partial}{\partial t} \wedge \Box_1 \qquad (\widetilde{\Lambda_2, \Box_2})_1 = \mathrm{e}^{-t}\Lambda_2 + \mathrm{e}^{-t}\frac{\partial}{\partial t} \wedge \Box_2$$

are compatible Poisson structures on the product manifold $M \times \mathbb{R}$.

Now, suppose that $(\Lambda_1, \Box_1)$ and $(\Lambda_2, \Box_2)$ are compatible Jacobi structures on a manifold M. Denote by Λ and by $\Box$ the 4-vector and the 3-vector on M given by

$$\Lambda = \Lambda_1 \wedge \Lambda_2 \qquad \Box = \Box_1 \wedge \Lambda_2 + \Box_2 \wedge \Lambda_1. \tag{15}$$

Using proposition 3.4(ii), we have

$$[\Lambda, \Lambda] = 4\Lambda \wedge \Box \qquad [\Lambda, \Box] = 0.$$

Thus, $(M, \Lambda, \frac{2}{3}\Box)$ is a generalized Jacobi manifold of order 4 (see proposition 2.3).

Remark 3.5. A direct computation proves that the 4-vector

$$(\widetilde{\Lambda, \Box})_2 = \mathrm{e}^{-2t}\Lambda + \mathrm{e}^{-2t}\frac{\partial}{\partial t} \wedge \Box$$

is just

$$(\widetilde{\Lambda, \Box})_2 = (\widetilde{\Lambda_1, \Box_1})_1 \wedge (\widetilde{\Lambda_2, \Box_2})_1.$$

Thus, from proposition 2.4, we deduce that $(M \times \mathbb{R}, (\widetilde{\Lambda, \Box})_2)$ is a generalized Poisson manifold of order four. In fact, if $\bar{\Lambda}_1$ and $\bar{\Lambda}_2$ are two compatible Poisson structures on a manifold N then $\bar{\Lambda}_1 \wedge \bar{\Lambda}_2$ is a generalized Poisson structure of order four on N (see [21]).

Next, we will describe a particular case, a generalized Jacobi structure on the phase space of a time-dependent Hamiltonian system.

As is well known, the phase space of a time-dependent Hamiltonian system is a product manifold $M = \mathbb{R} \times T^*Q$, where T^*Q is the cotangent bundle of a differentiable manifold Q of dimension m. Denote by λ_Q the Liouville 1-form of T^*Q and by $\Omega_Q = -\mathrm{d}\lambda_Q$ the canonical symplectic structure on T^*Q (see [30] for details).

If $\mathrm{pr}_1 : \mathbb{R} \times T^*Q \longrightarrow \mathbb{R}$ and $\mathrm{pr}_2 : \mathbb{R} \times T^*Q \longrightarrow T^*Q$ are the canonical projections onto the first and second factor, respectively, then a direct computation proves that

$$\eta_1 = \mathrm{pr}_1^*(\mathrm{d}t) - \mathrm{pr}_2^*(\lambda_Q)$$

is a contact 1-form on $\mathbb{R} \times T^*Q$. In fact, if $(q^1, \ldots, q^m, p_1, \ldots, p_m)$ are fibred coordinates on T^*Q then

$$\eta_1 = \mathrm{d}t - \sum_{i=1}^{m} p_i\, \mathrm{d}q^i.$$

Thus, if $(\Lambda_1, \Box_1)$ is the associated Jacobi structure with the contact 1-form η_1, we have

$$\Lambda_1 = \sum_{i=1}^{m}\left(\frac{\partial}{\partial q^i} + p_i\frac{\partial}{\partial t}\right) \wedge \frac{\partial}{\partial p_i} \qquad \Box_1 = \frac{\partial}{\partial t}. \tag{16}$$

On the other hand, we also can define in $\mathbb{R} \times T^*Q$ a Poisson structure Λ_2 as follows. Let $f \in C^\infty(\mathbb{R} \times T^*Q, \mathbb{R})$. Then, we consider the vector field X^2_f which is characterized by the following conditions:

$$(\mathrm{pr}_1^*(\mathrm{d}t))(X^2_f) = 0 \qquad i_{X^2_f}(\mathrm{pr}_2)^*(\Omega_Q) = \mathrm{d}f - \frac{\partial f}{\partial t}\mathrm{pr}_1^*(\mathrm{d}t).$$

Now, we define the bracket of two functions f and g by

$$\{f, g\}_2 = \mathrm{pr}_2^*(\Omega_Q)(X^2_f, X^2_g).$$

Thus, $(\mathbb{R} \times T^*Q, \{\,,\,\}_2)$ is a Poisson manifold (see, for instance, [1, 11]). Moreover, if Λ_2 is the associated Poisson structure then the Hamiltonian vector field $X^{\Lambda_2}_f$ of a function f is X^2_f. The structure Λ_2 is just the Poisson structure on $\mathbb{R} \times T^*Q$ defined by the canonical cosymplectic structure of $\mathbb{R} \times T^*Q$ (for the definition and properties of cosymplectic manifolds, see [1, 11, 30]).

In fibred coordinates $(t, q^1, \ldots, q^m, p_1, \ldots, p_m)$, we have

$$\Lambda_2 = \sum_{i=1}^{m} \frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial p_i}. \tag{17}$$

Using (16), (17) and proposition 3.4(ii), we deduce that the Jacobi structure $(\Lambda_1, \Box_1)$ and the Poisson structure Λ_2 are compatible. Therefore, if Λ and $\Box$ are the 4-vector and the 3-vector on $\mathbb{R} \times T^*Q$ given by

$$\Lambda = \Lambda_1 \wedge \Lambda_2 = \sum_{i,j} \left(\frac{\partial}{\partial q^i} + p_i \frac{\partial}{\partial t}\right) \wedge \frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial q^j} \wedge \frac{\partial}{\partial p_j}$$

$$\Box = \Box_1 \wedge \Lambda_2 = \sum_i \frac{\partial}{\partial t} \wedge \frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial p_i}$$

then we conclude that the pair $(\Lambda, \frac{2}{3}\Box)$ defines a generalized Jacobi structure of order four on $\mathbb{R} \times T^*Q$.

Example 6. (*3-Sasakian manifolds*) An interesting example of generalized Jacobi manifold are the 3-Sasakian manifolds [26] (see also [6, 7, 10, 15]).

First, we recall the definition of hyper-Kähler manifold which will be useful in the following.

A *hyper-Kähler manifold* is a differentiable manifold M of dimension $4n$ with a Riemannian metric g and three complex structures J_1, J_2, J_3 compatible with g and such that:

(i) the complex structures satisfy the quaternionic relations, i.e. $J_3 = J_1 \circ J_2 = -J_2 \circ J_1$;

(ii) the Kähler forms Ω_i, defined by $\Omega_i(X, Y) = g(X, J_iY)$ for all $X, Y \in \mathfrak{X}(M)$, are closed.

If we consider the Poisson structure $\bar{\Lambda}_i$ associated with the Kähler form Ω_i, then $[\bar{\Lambda}_i, \bar{\Lambda}_j] = 0$ for all i, j, that is, $\bar{\Lambda}_1$, $\bar{\Lambda}_2$ and $\bar{\Lambda}_3$ are compatible (see [21]). Thus, the fundamental 4-vector

$$\bar{\Lambda} = \bar{\Lambda}_1 \wedge \bar{\Lambda}_1 + \bar{\Lambda}_2 \wedge \bar{\Lambda}_2 + \bar{\Lambda}_3 \wedge \bar{\Lambda}_3$$

defines a generalized Poisson structure of order 4 on M (see [21]).

A 3-*Sasakian manifold* is a $(4n + 3)$-dimensional Riemannian manifold (M, g) that admits three distinct Sasakian structures $(\varphi_i, \Box_i, \eta_i, g)_{i=1,2,3}$ whose vector fields $\Box_1, \Box_2, \Box_3$ are mutually orthogonal and satisfy $[\Box_{\sigma(1)}, \Box_{\sigma(2)}] = 2\varepsilon_\sigma \Box_{\sigma(3)}$, for all $\sigma \in \mathfrak{S}_3$ (see [10, 26]).

If $(\varphi_i, \Box_i, \eta_i, g)_{i=1,2,3}$ is a 3-Sasakian structure on M then it can be proved (see [26]) that $(\varphi_i, \Box_i, \eta_i, g)_{i=1,2,3}$ is an almost contact 3-structure, that is

$$\begin{gathered}\eta_{\sigma(1)}(\Box_{\sigma(2)}) = 0 \qquad \varphi_{\sigma(1)}\Box_{\sigma(2)} = \varepsilon_\sigma \Box_{\sigma(3)} \qquad \eta_{\sigma(1)} \circ \varphi_{\sigma(2)} = \varepsilon_\sigma \eta_{\sigma(3)} \\ \varphi_{\sigma(1)} \circ \varphi_{\sigma(2)} = \eta_{\sigma(2)} \otimes \Box_{\sigma(1)} + \varepsilon_\sigma \varphi_{\sigma(3)}\end{gathered} \tag{18}$$

for all $\sigma \in \mathfrak{S}_3$.

Proposition 3.6. Let $(\varphi_i, \Box_i, \eta_i, g)_{i=1,2,3}$ be a 3-Sasakian structure on a manifold M and suppose that $(\Lambda_i, \Box_i)$ is the associated Jacobi structure with the contact 1-form η_i, $i = 1, 2, 3$. Then we have that:

(i) the structures $(\Lambda_1, \Box_1)$, $(\Lambda_2, \Box_2)$ and $(\Lambda_3, \Box_3)$ are compatible;

(ii) if Λ and $\Box$ are the 4-vector and the 3-vector on M given by

$$\Lambda = \sum_{i=1}^{3} \Lambda_i \wedge \Lambda_i \qquad \Box = \sum_{i=1}^{3} \Box_i \wedge \Lambda_i$$

then $(\Lambda, \frac{2}{3}\Box)$ is a generalized Jacobi structure on M of order four.

Proof. Using (18) and remark 2.5, we obtain that $(M \times \mathbb{R}, J_1, J_2, J_3, h)$ is a hyper-Kähler manifold, where

$$J_i = \varphi_i \circ (\mathrm{pr}_1)_* - \frac{1}{2}(\mathrm{pr}_2)^*(\mathrm{d}t) \otimes \Box_i + 2(\mathrm{pr}_1)^*(\eta_i) \otimes \frac{\partial}{\partial t}$$
$$h = \mathrm{e}^t(2\mathrm{pr}_1^*(g) + \frac{1}{2}\mathrm{pr}_2^*(\mathrm{d}t) \otimes \mathrm{pr}_2^*(\mathrm{d}t))$$

and $\mathrm{pr}_1 : M \times \mathbb{R} \longrightarrow M$ and $\mathrm{pr}_2 : M \times \mathbb{R} \longrightarrow \mathbb{R}$ denote the canonical projections onto the first and second factor, respectively.

Now, let Ω_i be the Kähler 2-form of the Kähler structure (J_i, h) and let $\bar{\Lambda}_i$ be the associated Poisson structure with Ω_i, $i = 1, 2, 3$. Since $(M \times \mathbb{R}, \Omega_i)$ is the symplectification of the contact manifold (M, η_i), we have that

$$\bar{\Lambda}_i = (\widetilde{\Lambda_i, \Box_i})_1 \qquad i = 1, 2, 3.$$

Thus, using proposition 3.4(iii), we deduce that the structures $(\Lambda_1, \Box_1)$, $(\Lambda_2, \Box_2)$ and $(\Lambda_3, \Box_3)$ are compatible. This proves (i).

(ii) follows from (i) and proposition 3.4(ii). $\Box$

We will describe an interesting particular case, a generalized Jacobi structure on the sphere S^{4n+3}.

On $\mathbb{R}^{4n+4}$, we consider the usual hyper-Kähler structure (J_1, J_2, J_3, h). Let Ω_i be the Kähler 2-form of the Hermitian structure (J_i, h), $i = 1, 2, 3$. Then the Poisson vectors $\bar{\Lambda}_1$, $\bar{\Lambda}_2$ and $\bar{\Lambda}_3$ associated with Ω_1, Ω_2 and Ω_3 respectively, are given by

$$\bar{\Lambda}_1 = \sum_{i=1}^{n+1} \left(\frac{\partial}{\partial x_i^2} \wedge \frac{\partial}{\partial x_i^1} + \frac{\partial}{\partial x_i^4} \wedge \frac{\partial}{\partial x_i^3} \right)$$
$$\bar{\Lambda}_2 = \sum_{i=1}^{n+1} \left(\frac{\partial}{\partial x_i^3} \wedge \frac{\partial}{\partial x_i^1} + \frac{\partial}{\partial x_i^2} \wedge \frac{\partial}{\partial x_i^4} \right)$$
$$\bar{\Lambda}_3 = \sum_{i=1}^{n+1} \left(\frac{\partial}{\partial x_i^4} \wedge \frac{\partial}{\partial x_i^1} + \frac{\partial}{\partial x_i^3} \wedge \frac{\partial}{\partial x_i^2} \right)$$

where $(x_i^1, x_i^2, x_i^3, x_i^4)$ are the usual coordinates on $\mathbb{R}^{4n+4}$.

On the other hand, let A be the radial vector field on $\mathbb{R}^{4n+4}$, that is

$$A = \sum_{j=1}^{4}\sum_{i=1}^{n+1} x_i^j \frac{\partial}{\partial x_i^j}.$$

Define the vector fields on $\mathbb{R}^{4n+4}$: $\bar{\Box}_1 = -J_1 A$, $\bar{\Box}_2 = -J_2 A$ and $\bar{\Box}_3 = -J_3 A$. Then, the restrictions $\Box_1$, $\Box_2$ and $\Box_3$ to S^{4n+3} of $\bar{\Box}_1$, $\bar{\Box}_2$ and $\bar{\Box}_3$, respectively, are tangent to S^{4n+3}.

Now, suppose that $\bar{\eta}_i$, $i = 1, 2, 3$, is the 1-form on $\mathbb{R}^{4n+4}$ defined by $\bar{\eta}_i(X) = h(X, \bar{\Box}_i)$ for $X \in \mathfrak{X}(\mathbb{R}^{4n+4})$. Thus,

$$\begin{aligned}
\bar{\eta}_1 &= \sum_{i=1}^{n+1}(x_i^2\,\mathrm{d}x_i^1 - x_i^1\,\mathrm{d}x_i^2 - x_i^3\,\mathrm{d}x_i^4 + x_i^4\,\mathrm{d}x_i^3)\\
\bar{\eta}_2 &= \sum_{i=1}^{n+1}(x_i^3\,\mathrm{d}x_i^1 - x_i^1\,\mathrm{d}x_i^3 - x_i^4\,\mathrm{d}x_i^2 + x_i^2\,\mathrm{d}x_i^4) \qquad (19)\\
\bar{\eta}_3 &= \sum_{i=1}^{n+1}(x_i^4\,\mathrm{d}x_i^1 - x_i^1\,\mathrm{d}x_i^4 - x_i^2\,\mathrm{d}x_i^3 + x_i^3\,\mathrm{d}x_i^2).
\end{aligned}$$

Consider the (1,1)-tensor field $\bar{\varphi}_i$ on $\mathbb{R}^{4n+4}$ given by

$$\bar{\varphi}_i = J_i - \bar{\eta}_i \otimes A \qquad i = 1, 2, 3.$$

If v is a tangent vector to S^{4n+3} at x then $(\bar{\varphi}_i)_x(v)$ is also tangent to S^{4n+3}. Therefore, the restriction φ_i of $\bar{\varphi}_i$ to S^{4n+3} is a (1,1)-tensor field on S^{4n+3}. Moreover, from the results of [26], we deduce that $(\varphi_i, \Box_i, \eta_i, g)_{i=1,2,3}$ is a 3-Sasakian structure on S^{4n+3}, where

$$g = j^* h \qquad \eta_i = j^*(\bar{\eta}_i)$$

$j : S^{4n+3} \hookrightarrow \mathbb{R}^{4n+4}$ being the canonical inclusion.

A direct computation proves that the restriction Λ_i to S^{4n+3} of the 2-vector $\Gamma_i = \frac{1}{2}(\bar{\Lambda}_i - A \wedge \bar{\Box}_i)$ is tangent to S^{4n+3}. In fact, using (6), we have that $(\Lambda_i, \Box_i)$ is the Jacobi structure on S^{4n+3} associated with the contact 1-form η_i.

Consequently, from proposition 3.6, we conclude that the generalized Jacobi structure of order four associated with the 3-Sasakian structure $(\varphi_i, \Box_i, \eta_i, g)_{i=1,2,3}$ is defined by the restriction of

$$\left(\frac{1}{4}\sum_i(\bar{\Lambda}_i \wedge \bar{\Lambda}_i) + \frac{1}{2}\sum_i(\bar{\Box}_i \wedge \bar{\Lambda}_i) \wedge A, \frac{1}{3}\sum_i(\bar{\Box}_i \wedge \bar{\Lambda}_i)\right)$$

to S^{4n+3}.

Remark 3.7. In the particular case when $n = 1$, the 1-forms given in (19) generate the basic instanton on S^7 (see [6]).

Example 7. (*Generalized Jacobi structure on the unit sphere of a generalized Lie algebra*) Let $\mathfrak{g}$ be a vector space of dimension m. A skew-symmetric Lie $2n$-bracket on $\mathfrak{g}$ is a $2n$-linear skew-symmetric mapping

$$[, \ldots,] : \mathfrak{g} \times \ldots^{(2n} \ldots \times \mathfrak{g} \longrightarrow \mathfrak{g}$$

satisfying the following condition:

$$\mathrm{Alt}[a_1, \ldots, a_{2n-1}, [b_1, \ldots, b_{2n}]] = 0 \qquad (20)$$

for all $a_1, \ldots, a_{2n-1}, b_1, \ldots, b_{2n} \in \mathfrak{g}$. A vector space endowed with such a bracket is called a *generalized Lie algebra of order* $2n$ (see [4]).

Now, on the dual space $\mathfrak{g}^*$, we define the generalized almost Poisson bracket of order $2n$

$$\{,\ldots,\}: C^\infty(\mathfrak{g}^*,\mathbb{R})\times\cdots\times C^\infty(\mathfrak{g}^*,\mathbb{R})\longrightarrow C^\infty(\mathfrak{g}^*,\mathbb{R})$$

by putting

$$\{f_1,\ldots,f_{2n}\}_\alpha=\alpha([(\mathrm{d}f_1)_\alpha,\ldots,(\mathrm{d}f_{2n})_\alpha]) \tag{21}$$

for all $f_1,\ldots,f_{2n}\in C^\infty(\mathfrak{g}^*,\mathbb{R})$ and $\alpha\in\mathfrak{g}^*$. Note that $(\mathrm{d}f_i)_\alpha\in T_\alpha\mathfrak{g}^*\cong(\mathfrak{g}^*)^*\cong\mathfrak{g}$.

Let $\bar\Lambda$ be the $2n$-vector associated with the bracket defined in (21). Then we have the following.

Proposition 3.8. $(\mathfrak{g}^*,\bar\Lambda)$ is a generalized Poisson manifold of order $2n$.

Proof. Let $\{e_i\}_{i=1,\ldots,m}$ be a basis of $\mathfrak{g}$. Denote by $\{e^i\}_{i=1,\ldots,m}$ the dual basis of $\mathfrak{g}^*$ and by $(x_1,\ldots,x_m)$ the corresponding global coordinates on $\mathfrak{g}^*$.

Suppose that $[e_{i_1},\ldots,e_{i_{2n}}]=\sum_j C^j_{i_1\ldots i_{2n}}e_j$, with $C^j_{i_1\ldots i_{2n}}\in\mathbb{R}$. From (21), we deduce that

$$\bar\Lambda=\frac{1}{(2n)!}\sum_{j,i_1,\ldots,i_{2n}}C^j_{i_1\ldots i_{2n}}x_j\frac{\partial}{\partial x_{i_1}}\wedge\cdots\wedge\frac{\partial}{\partial x_{i_{2n}}}. \tag{22}$$

Moreover, a straightforward computation, using (20), proves that $[\bar\Lambda,\bar\Lambda]=0$. Hence $(\mathfrak{g}^*,\bar\Lambda)$ is a generalized Poisson manifold of order $2n$. $\square$

Now, let $\langle,\rangle$ be an inner product on $\mathfrak{g}$ and g the corresponding Riemannian metric. Note that, in this case, we can identify $\mathfrak{g}$ with $\mathfrak{g}^*$. Consequently, via this identification, we have that the $2n$-vector $\bar\Lambda$ induces a generalized Poisson structure on $\mathfrak{g}$ which we also denote by $\bar\Lambda$.

Consider $\{e_i\}_{i=1,\ldots,m}$ an orthonormal basis of $\mathfrak{g}$ and $(x_1,\ldots,x_m)$ the corresponding global coordinates on $\mathfrak{g}$. If A is the radial vector field on $\mathfrak{g}$ and σ is the 1-form on $\mathfrak{g}$ given by $\sigma(X)=g(X,A)$, for $X\in\mathfrak{X}(\mathfrak{g})$, then we have

$$A=\sum_{i=1}^m x_i\frac{\partial}{\partial x_i}\qquad\text{and}\qquad\sigma=\sum_{i=1}^m x_i\,\mathrm{d}x_i. \tag{23}$$

From (22) and (23), we obtain that

$$\mathcal{L}_A\bar\Lambda=-(2n-1)\bar\Lambda. \tag{24}$$

Denote by $S^1(\mathfrak{g})=\{a\in\mathfrak{g}/\|a\|=1\}$ the unit sphere in $\mathfrak{g}$. Next, we will obtain a generalized Jacobi structure on $S^1(\mathfrak{g})$.

Proposition 3.9. Let $\mathfrak{g}$ be a generalized Lie algebra of order $2n$ endowed with an inner product $\langle,\rangle$. Suppose that $\bar\Lambda$ is the corresponding generalized Poisson structure on $\mathfrak{g}$. Then, the restrictions Λ and $\Box$ to $S^1(\mathfrak{g})$ of the $2n$-vector $\bar\Lambda'$ and the $(2n-1)$-vector $\bar\Box'$ on $\mathfrak{g}$ defined by

$$\bar\Lambda'=\bar\Lambda-A\wedge i_\sigma\bar\Lambda\qquad\bar\Box'=i_\sigma\bar\Lambda$$

are tangent to $S^1(\mathfrak{g})$. Moreover, the pair $(\Lambda,\Box)$ defines a generalized Jacobi structure of order $2n$ on $S^1(\mathfrak{g})$.

Proof. If we identify the open subset $\mathfrak{g} - \{0\}$ of $\mathfrak{g}$ with the product manifold $S^1(\mathfrak{g}) \times \mathbb{R}$ via the diffeomorphism

$$F : \mathfrak{g} - \{0\} \longrightarrow S^1(\mathfrak{g}) \times \mathbb{R} \qquad a \mapsto \left(\frac{a}{\|a\|}, \ln \|a\|\right)$$

then we get

$$F_*(A_{|\mathfrak{g}-\{0\}}) = \frac{\partial}{\partial t} \tag{25}$$

where t is the usual coordinate on $\mathbb{R}$ and $A_{|\mathfrak{g}-\{0\}}$ is the restriction of the radial vector field A to $\mathfrak{g} - \{0\}$. Moreover, the $2n$-vector

$$\underline{\Lambda} = F_*(\bar{\Lambda}_{|\mathfrak{g}-\{0\}}) \tag{26}$$

defines a generalized Poisson structure on $S^1(\mathfrak{g}) \times \mathbb{R}$ and thus, $(S^1(\mathfrak{g}) \times \mathbb{R}, \underline{\Lambda})$ is a generalized Poisson manifold of order $2n$.

Now, we consider the generalized almost Jacobi structure $(\underline{\Lambda}', \underline{\Box}')$ of order $2n$ on $S^1(\mathfrak{g}) \times \mathbb{R}$ given by

$$\underline{\Box}' = \mathrm{e}^{(2n-1)t} i_{\mathrm{d}t} \underline{\Lambda} \qquad \text{and} \qquad \underline{\Lambda}' = \mathrm{e}^{(2n-1)t} \underline{\Lambda} - \frac{\partial}{\partial t} \wedge \underline{\Box}'. \tag{27}$$

Using (24), (25) and (27), we deduce that

$$i_{\mathrm{d}t} \underline{\Box}' = 0 \qquad \mathcal{L}_{\partial/\partial t} \underline{\Box}' = 0 \qquad i_{\mathrm{d}t} \underline{\Lambda}' = 0 \text{ and } \mathcal{L}_{\partial/\partial t} \underline{\Lambda}' = 0$$

which implies that $\underline{\Lambda}'$ and $\underline{\Box}'$ induce a $2n$-vector Λ' and a $(2n-1)$-vector $\Box'$ on $S^1(\mathfrak{g})$. On the other hand, since $\widetilde{(\Lambda', \Box')}_{2n-1} = \underline{\Lambda}$, we conclude, from proposition 2.4, that $(\Lambda', \Box')$ is a generalized Jacobi structure of order $2n$ on $S^1(\mathfrak{g})$.

Therefore, the result follows using (25)–(27) and the following facts

$$F(S^1(\mathfrak{g})) = S^1(\mathfrak{g}) \times \{0\} \qquad F^*(\mathrm{d}t) = \frac{i^*(\sigma)}{\|i^*(\sigma)\|}$$

where $i : \mathfrak{g} - \{0\} \hookrightarrow \mathfrak{g}$ is the canonical inclusion. $\Box$

Remark 3.10. For $n = 1$, we recover the Jacobi structure on $S^1(\mathfrak{g})$ obtained by Lichnerowicz in [33].

Next, we will describe the interesting particular case of a generalized Jacobi structure on the unit sphere of a simple compact Lie algebra $\mathfrak{g}$. First, we will recall the definition of the skew-symmetric Lie $2n$-bracket on $\mathfrak{g}$ introduced in [3].

Let $(\mathfrak{g}, [\ ,\ ])$ be the Lie algebra of a simple compact Lie group G. If B is the Killing form of $\mathfrak{g}$ then $-B$ defines an inner product $\langle\ ,\ \rangle$ on $\mathfrak{g}$. We choose an orthonormal basis $\{e_i\}_{i=1,\ldots,m}$ of $\mathfrak{g}$ and denote by $(x_1, \ldots, x_m)$ the corresponding global coordinates on $\mathfrak{g}$.

If $K : \mathfrak{g} \times \ldots^{(n+1} \ldots \times \mathfrak{g} \to \mathbb{R}$ is a primitive invariant symmetric polynomial on $\mathfrak{g}$, we consider the skew-symmetric multilineal tensor $w : \mathfrak{g} \times \ldots^{(2n+1} \ldots \times \mathfrak{g} \longrightarrow \mathbb{R}$ given by

$$w(e_{i_1}, \ldots, e_{i_{2n+1}}) = w_{i_1 \ldots i_{2n+1}} = \sum_{\sigma \in \mathfrak{S}_{2n+1}} \varepsilon_\sigma K([e_{\sigma(i_1)}, e_{\sigma(i_2)}], \ldots, [e_{\sigma(i_{2n-1})}, e_{\sigma(i_{2n})}], e_{\sigma(i_{2n+1})})$$

where ε_σ is the signature of the permutation σ.

Then, using the inner product $\langle\ ,\ \rangle$, we can identify upper and lower indices which allow us to obtain a skew-symmetric Lie $2n$-bracket $[, \ldots,] : \mathfrak{g} \times \ldots^{(2n} \ldots \times \mathfrak{g} \longrightarrow \mathfrak{g}$ on $\mathfrak{g}$ defined by (see [3])

$$[e_{i_1}, \ldots, e_{i_{2n}}] = \sum_{j=1}^{m} w^j_{i_1 \ldots i_{2n}} e_j. \tag{28}$$

Therefore, from (22), (23), (28) and proposition 3.9, we deduce that the restrictions to $S^1(\mathfrak{g})$ of the $2n$-vector $\bar{\Lambda}'$ and the $(2n-1)$-vector $\bar{\Box}'$ on $\mathfrak{g}$ given by

$$\bar{\Lambda}' = \frac{1}{(2n)!} \sum_{j,i_1,\ldots,i_{2n}} \left(w^j_{i_1\ldots i_{2n}} - \sum_{s=1}^{2n} \sum_{k=1}^{m} w^j_{i_1\ldots i_{s-1} k i_{s+1}\ldots i_{2n}} x_k x_{i_s} \right) x_j \frac{\partial}{\partial x_{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial x_{i_{2n}}}$$

$$\bar{\Box}' = \frac{1}{(2n)!} \sum_{j,i_1,\ldots,i_{2n}} \sum_{k=1}^{2n} (-1)^{k+1} w^j_{i_1\ldots i_{2n}} x_j x_{i_k} \frac{\partial}{\partial x_{i_1}} \wedge \cdots \wedge \widehat{\frac{\partial}{\partial x_{i_k}}} \wedge \cdots \wedge \frac{\partial}{\partial x_{i_{2n}}}$$

define a generalized Jacobi structure $(\Lambda, \Box)$ of order $2n$ on $S^1(\mathfrak{g})$.

4. Hamiltonian vector fields on a generalized almost Jacobi manifold

First, we will introduce the definition of a Hamiltonian vector field on a generalized almost Jacobi manifold.

If $(M, \Lambda, \Box)$ is a generalized almost Jacobi manifold of order n, then (M, Λ) and $(M, \Box)$ are generalized almost Poisson manifolds of order n and $n-1$, respectively, and we will denote by $X^{\Lambda}_{f_1\ldots f_{n-1}}$ (respectively, $X^{\Box}_{f_1\ldots\widehat{f_i}\ldots f_{n-1}}$) the Hamiltonian vector field on (M, Λ) (respectively, $(M, \Box)$) associated with the functions $f_1, \ldots, f_{n-1}$ (respectively, $f_1, \ldots, f_{i-1}, f_{i+1}, \ldots, f_{n-1}$).

Definition 4.1. Let $(M, \Lambda, \Box)$ be a generalized almost Jacobi manifold of order n. Suppose that $f_1, \ldots, f_{n-1}$ are C^∞ real-valued functions on M. Then the vector field $X_{f_1\ldots f_{n-1}}$ on M given by

$$X_{f_1\ldots f_{n-1}} = X^{\Lambda}_{f_1\ldots f_{n-1}} + \sum_{i=1}^{n-1} (-1)^{i-1} f_i X^{\Box}_{f_1\ldots\widehat{f_i}\ldots f_{n-1}} \tag{29}$$

is called the Hamiltonian vector field associated with the functions $f_1, \ldots, f_{n-1}$.

Remark 4.2. Note that the Hamiltonian vector field $X_{1 f_1\ldots f_{n-2}}$ is just the vector field $X^{\Box}_{f_1\ldots f_{n-2}}$.

Now, for every point x of a generalized almost Jacobi manifold $(M, \Lambda, \Box)$ of order n we consider the subspace $\mathcal{D}_x$ of $T_x M$ generated by all the Hamiltonian vector fields evaluated at the point x. Thus, we obtain a generalized distribution $\mathcal{D}$ on M which will be called the *characteristic distribution*.

If $n = 2$ and M is a Jacobi manifold, $\mathcal{D}$ is involutive so that it defines a generalized foliation on M in the sense of Sussmann [40]. Moreover, if L is a leaf of the characteristic foliation then the Jacobi structure $(\Lambda, \Box)$ induces a Jacobi structure on L, and L with the induced structure is a contact manifold or a l.c.s. manifold (for a detailed study we refer to [13]).

The things are drastically different for $n \geqslant 3$. In fact, there exist examples of generalized Poisson manifolds of order $n \geqslant 3$ such that their characteristic distributions are not involutive (see [19]).

Next, we will give an example of a generalized Jacobi manifold of order $n \geqslant 3$ such that its characteristic distribution is not involutive.

Example 8. Let $\bar{\Lambda}$ be the 4-vector on $\mathbb{R}^5$ given by

$$\bar{\Lambda} = x_4 \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial x_4} + \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial x_4} \wedge \frac{\partial}{\partial x_5}$$

where $(x_1, x_2, x_3, x_4, x_5)$ are the standard coordinates on $\mathbb{R}^5$. Then $\bar{\Lambda}$ defines a generalized Poisson structure of order 4 on $\mathbb{R}^5$ (note that $[\bar{\Lambda}, \bar{\Lambda}] = 0$).

A direct computation proves that

$$\begin{aligned}
&X^{\bar{\Lambda}}_{x_1x_2x_3} = x_4 \frac{\partial}{\partial x_4} \qquad X^{\bar{\Lambda}}_{x_1x_2x_4} = -x_4 \frac{\partial}{\partial x_3} \qquad X^{\bar{\Lambda}}_{x_1x_2x_5} = 0 \\
&X^{\bar{\Lambda}}_{x_1x_3x_4} = x_4 \frac{\partial}{\partial x_2} \qquad X^{\bar{\Lambda}}_{x_1x_3x_5} = 0 \qquad X^{\bar{\Lambda}}_{x_1x_4x_5} = 0 \\
&X^{\bar{\Lambda}}_{x_2x_3x_4} = -x_4 \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_5} \qquad X^{\bar{\Lambda}}_{x_2x_3x_5} = -\frac{\partial}{\partial x_4} \\
&X^{\bar{\Lambda}}_{x_2x_4x_5} = \frac{\partial}{\partial x_3} \qquad X^{\bar{\Lambda}}_{x_3x_4x_5} = -\frac{\partial}{\partial x_2}.
\end{aligned} \tag{30}$$

Consider the generalized almost Jacobi structure $(\Lambda, \Box)$ on $\mathbb{R}^5$ given by

$$\begin{aligned}
\Lambda &= x_3\bar{\Lambda} = x_3x_4 \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial x_4} + x_3 \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial x_4} \wedge \frac{\partial}{\partial x_5} \\
\Box &= \frac{1}{3} i_{\mathrm{d}x_3} \bar{\Lambda} = \frac{1}{3} x_4 \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_4} - \frac{1}{3} \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_4} \wedge \frac{\partial}{\partial x_5}.
\end{aligned}$$

Using the results of section 3 (see example 4) we deduce that $(\Lambda, \Box)$ defines a generalized Jacobi structure on $\mathbb{R}^5$ of order four. From (29) we obtain that the Hamiltonian vector field on $(M, \Lambda, \Box)$ associated with the functions x_i, x_j, x_k is given by

$$X_{x_ix_jx_k} = x_3 X^{\bar{\Lambda}}_{x_ix_jx_k} + \tfrac{1}{3} x_i X^{\bar{\Lambda}}_{x_jx_kx_3} - \tfrac{1}{3} x_j X^{\bar{\Lambda}}_{x_ix_kx_3} + \tfrac{1}{3} x_k X^{\bar{\Lambda}}_{x_ix_jx_3}. \tag{31}$$

Using (30) and (31) we deduce that the vector field

$$[X_{1x_1x_2}, X_{1x_2x_4}] = \frac{1}{9} x_4 \frac{\partial}{\partial x_1}$$

does not belong to $\mathcal{D}$.

If (M, Λ) is a Nambu–Poisson manifold of order n, and $\mathcal{D}$ is its characteristic distribution, then $\mathcal{D}$ defines a generalized foliation on M whose leaves are either points or n-dimensional manifolds endowed with a Nambu–Poisson structure coming from a volume form (see [19]; see also [44]).

Now, we will show that the characteristic distribution of a Nambu–Jacobi manifold is also completely integrable.

Theorem 4.3. The characteristic distribution of a Nambu–Jacobi manifold M defines a generalized foliation on M.

Proof. Let $(M, \Lambda, \Box)$ be a m-dimensional Nambu–Jacobi manifold of order n with Nambu–Jacobi bracket $\{, \ldots, \}$. If $f_1, \ldots, f_{n-1}, h \in C^\infty(M, \mathbb{R})$ we deduce from (29) that

$$X_{f_1 \cdots f_{n-1}}(h) = \{f_1, \ldots, f_{n-1}, h\} + (-1)^n h \Box(\mathrm{d}f_1, \ldots, \mathrm{d}f_{n-1}). \tag{32}$$

Using (4) and (32) we have

$$[X_{f_1\ldots f_{n-1}}, X_{g_1\ldots g_{n-1}}] = \sum_{i=1}^{n-1} X_{g_1\ldots g_{i-1}\{f_1,\ldots,f_{n-1},g_i\}g_{i+1}\ldots g_{n-1}}$$

which implies that the characteristic distribution $\mathcal{D}$ is involutive.

Moreover, if $x \in M$ and $(x_1, \ldots, x_m)$ is a local coordinate system around x then the distribution $\mathcal{D}$ is locally generated by the Hamiltonian vector fields $X_{x_{i_1}\ldots x_{i_{n-1}}}$ and $X_{1x_{i_1}\ldots x_{i_{n-2}}}$, with $1 \leqslant i_1 < \cdots < i_{n-1} \leqslant m$. Thus, using a result of Lecomte [27] (see also [18], p 414), we conclude that $\mathcal{D}$ is invariant. This ends the proof of our result. □

Now, we will see that the leaves of the characteristic foliation of a Nambu–Jacobi manifold have an induced Nambu–Jacobi structure and we will study the nature of such a structure.

Theorem 4.4. Let $(M, \Lambda, \Box)$ be a Nambu–Jacobi manifold of order n with $n \geqslant 3$ and $x \in M$. Suppose that $\mathcal{D}$ is the characteristic foliation of M and that L is the leaf of $\mathcal{D}$ passing through x. Then $(\Lambda, \Box)$ induces a Nambu–Jacobi structure $(\Lambda_L, \Box_L)$ on L and we have:

(i) if $\Lambda(x) \neq 0$ then L has dimension n, Λ_L is the Nambu–Poisson structure of order n associated with a volume form on L, and there exists a closed 1-form θ_L on L such that $\Box_L = i_{\theta_L}\Lambda_L$;

(ii) if $\Lambda(x) = 0$ and $\Box(x) \neq 0$, then L has dimension $n-1$ and $\Lambda_L = 0$; moreover, we have:

(a) if $n > 3$ then $\Box_L$ is the Nambu–Poisson structure of order $n-1$ associated with a volume form on L;

(b) if $n = 3$ then $\Box_L$ is a symplectic structure on L;

(iii) if $\Lambda(x) = 0$ and $\Box(x) = 0$ then $L = \{x\}$ and the induced Nambu–Jacobi structure is trivial.

Proof. If we take n functions $f_1, \ldots, f_n$ defined on L, a bracket $\{f_1, \ldots, f_n\}_L$ can be defined as follows. We extend each f_j, $1 \leqslant j \leqslant n$, to a function $\tilde{f}_j$ on M, and put

$$\{f_1, \ldots, f_n\}_L(y) = \{\tilde{f}_1, \ldots, \tilde{f}_n\}(y)$$

for all $y \in L$. Since

$$\{\tilde{f}_1, \ldots, \tilde{f}_n\} = X_{\tilde{f}_1\ldots\tilde{f}_{n-1}}(\tilde{f}_n) - (-1)^n \tilde{f}_n X_{1\tilde{f}_1\ldots\tilde{f}_{n-2}}(\tilde{f}_{n-1})$$

(see (32) and remark 4.2) we deduce that $\{f_1, \ldots, f_n\}_L$ is independent on the chosen extensions $\tilde{f}_i$. Of course, $\{,\ldots,\}_L$ satisfies the fundamental identity. Thus, $\{,\ldots,\}_L$ is a Nambu–Jacobi bracket on L. Furthermore, it is clear that the structure $(\Lambda, \Box)$ restricts to L, and that the restriction is just the Nambu–Jacobi structure $(\Lambda_L, \Box_L)$ on L associated with the Nambu–Jacobi bracket $\{,\ldots,\}_L$.

Next, we will prove (i), (ii) and (iii).

(i) If $\Lambda(x) \neq 0$ then, from proposition 2.1, we deduce that there exists $\theta_x \in T_x^*M$ such that

$$i_{\theta_x}\Lambda(x) = \Box(x). \tag{33}$$

Moreover, since Λ is a Nambu–Poisson tensor (see proposition 2.1) then $\Lambda(x)$ is decomposable. Therefore, using (29) and (33), we deduce that the dimension of $\mathcal{D}_x$ is n. Hence, the dimension of L is n. This fact implies $\Lambda(y) \neq 0$ for all $y \in L$ and, consequently Λ_L is the Nambu–Poisson structure of order n associated with a volume form on L (see proposition II.8 in [19]).

On the other hand, the n-vector Λ_L induces an isomorphism of $C^\infty(L, \mathbb{R})$-modules $\#_L : \Omega^1(L) \longrightarrow \mathcal{V}^{n-1}(L)$ given by $\#_L(\alpha) = i_\alpha \Lambda_L$, for all $\alpha \in \Omega^1(L)$. Thus, there exists a unique 1-form θ_L on L such that $\Box_L = \#_L(\theta_L) = i_{\theta_L}\Lambda_L$ and, from proposition 3.2, we conclude that θ_L is closed.

(ii)(a) If $n > 3$, $\Lambda(x) = 0$ and $\Box(x) \neq 0$, then, proceeding as in the above case, we prove that the dimension of $\mathcal{D}_x$ is $n-1$. Therefore, the dimension of the leaf L is $n-1$ which implies that $\Lambda_L = 0$ and that $\Box_L(y) \neq 0$, for all $y \in L$. Thus, $\Box_L$ is the Nambu–Poisson structure of order $n-1$ associated with a volume form on L (see proposition II.8 in [19]).

(ii)(b) If $n = 3$, $\Lambda(x) = 0$ and $\Box(x) \neq 0$ then, since $\Box$ is a Poisson structure, we deduce that the dimension of $\mathcal{D}_x$ is even (see (29)). Hence, the dimension of L is also even. Consequently, $\Lambda_L = 0$ and $\Box_L$ is a symplectic structure on L (note that if $\Lambda(y) \neq 0$ for some point y of L then we would obtain that the dimension of L is three which is a contradiction).

Next, we will show that the dimension of L is two. For this purpose, we will prove that $\Box_L(x)$ is decomposable. Since $\Box_L(y) \neq 0$ for every point $y \in L$, we deduce that there exists an open neighbourhood U_L in L of x such that

$$\Box_L(\mathrm{d}f_1, \mathrm{d}f_2) \neq 0 \tag{34}$$

along U_L, where $f_i : U_L \to \mathbb{R}$ are C^∞ real-valued functions on U_L.

On the other hand, using (4) and the fact that $\Lambda_L = 0$, it follows that

$$\Box_L(\mathrm{d}f_1, \mathrm{d}f_2)\Box_L = X_{f_1}^{\Box_L} \wedge X_{f_2}^{\Box_L} \tag{35}$$

where $X_{f_i}^{\Box_L}$ is the Hamiltonian vector field on the symplectic manifold $(L, \Box_L)$ associated with the function f_i. Thus, from (34) and (35), we conclude that $\Box_L(x)$ is decomposable.

(iii) If $\Lambda(x) = 0$ and $\Box(x) = 0$ then all the Hamiltonian vector fields vanish at x. Hence, the leaf L reduces to the point x. $\Box$

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